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# Generalized thermoelastic-diffusion model with higher-order fractional time-derivatives and four-phase-lags

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**ABSTRACT**

The present work is devoted to the derivation of fundamental equations in generalized thermoelastic diffusion with four lags and higher-order time-fractional derivatives. The equations of the heat conduction and the mass diffusion have been modified by using Taylor's series of time-fractional order. In this new model, the Fourier and the Fick laws have been modified to include a higher time-fractional order of the heat conduction vector, the gradient of temperature, the diffusing mass flux and the gradient of chemical potential. We adopted the definitions of Caputo and Jumarie; for time-fractional derivatives. The work of Nowacki; Sherief, Hamza, and Saleh; and Aouadi; are deduced as limit cases from the current investigation. Applying this formulation, we have discussed a thermoelastic-diffusion problem for a half-space exposed to thermal and chemical shock with a permeable material in contact with the half-surface. We discussed the sensitivity of the different physical parameters in all studied fields in detail and the results are presented graphically as well as in tabular forms.

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Thermoelasticity; fractional heat conduction fractional Fickian diffusion; four-phase-lags; higher-order

## 1. Introduction

In recent years, the subject fractional calculus has been applied in an increasing number of fields, such as control engineering, electromagnetism, chemistry, signal processing, quantum mechanics, astrophysics, quantum field theory and nuclear physics, etc. The models of abnormal transport processes in the form of space and/or time-fractional convection-diffusion equation have been of great interest and have been studied by several researchers. Moreover, global dependency and non-local property of the fractional derivative is one of the main reasons for its increasing popularity. The concept of non-locality using the fractional differential operator in thermoelastic models opens up a new perspective on the study of thermoelastic deformations in solid mechanics. In addition, nonstandard constructions are necessary for continuous improvement in the progress of new materials.

The concepts of fractional calculus have been generalized in several approaches and some alternative definitions of fractional derivatives have been clarified in Podlubny (1999), Oldham and Spanier (1974), Miller and Ross (1993), Herrmann (2011), and Hilfer (2000). Caputo (1967) and Caputo and Mainardi (1971a, 1971b) have used the derivatives with fractional order to

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describe viscoelastic materials, and they successfully established the relationship between the linear viscoelasticity theory and fractional-order derivatives. They also attained an agreement with the experimental results successfully. Some applications of fractional calculus are analyzed for different problems of solid mechanics in Mainardi's book (Mainardi 1998).

The classical thermoelasticity theory containing an infinite speed of heat propagation contradicts physical realities. Over the last four decades, non-classical theories have been developed that include a limited speed of heat transfer in thermoelastic solids to remove this absurdity. The generalized thermoelasticity theory with relaxation time was first introduced by Lord and Shulman (1967), who attained a heat equation by proposing a new heat-conduction law instead of classical Fourier's law. One can refer to Hetnarski and Ignaczak (1999), to review and present the generalized theories of thermoelasticity.

We can define Diffusion as a random walk of assemblies of molecules from a high concentration region to a low concentration region. At present, there is great interest in studying this phenomenon due to its application in the electronics industry and geophysics. In the synthesis of the integrated circuit fabrication, we diffusion to introduce dopants in controlled quantities in the semiconductor material.

In fact, the improvement of advanced technologies in the years before, during, and after World War II has clearly affected investigations in which the fields of diffusion and temperature in solid materials cannot be ignored. In low and high temperatures, the heat and mass transfer processes play a critical role in many satellite problems, returning spacecraft, and landing on water or land. Nowadays, oil corporations are concerned with the thermo-diffusion process to extract oil more efficiently than oil deposits. Thermo-diffusion has many industrial applications like the optimal extraction of the oil from hydrocarbon reservoirs, fabrication of semiconductor devices in mixtures metal and molten semiconductor, separation of types like polymers and the manipulation of the macro-molecules like DNA, etc.

Nowacki (1974a, 1974b) developed the classical thermoelastic diffusion theory based on coupled thermoelastic theory. Olesiak and Pyryev (1995) investigated the coupled quasi-stationary thermo-diffusion problem of the elastic cylinder. The generalized theory of thermoelastic diffusion with a relaxation time has been established by Sherief, Hamza, and Saleh (2004), which allows finite speeds of the propagation of thermal waves. Also, they have demonstrated the theorems of reciprocity and uniqueness for the generalized equations of the thermo-diffusion problem, in an isotropic media. Based on this theory, Sherief and Saleh (2005) studied the problem of thermo-diffusion half-space.

Aouadi (2007) proved the theory of thermo-diffusion in the Laplace transformation, assuming that the field variables are continuous and that Laplace's inversion to each function is unique. Aouadi (2008) obtained the theories of reciprocity and uniqueness for the generalized thermo-diffusion problem in the anisotropic materials, with the restriction that the thermal conductivity, diffusion and elastic tensors were positive definite. More recently, the influences of thermodiffusionin anisotropic devices have been considered by many investigations (Abbas 2015; Deswal, Kalkal, and Sheoran 2016; Mishra, Sharma, and Sharma 2017; Othman and Eraki 2017; Xiong and Niu 2017; Biswas, Mukhopadhyay, and Shaw 2019; Davydov, Zemskov, and Akhmetova 2019; Mondal, Sur, and Kanoria 2019; Mondal and Kanoria 2019).

Recently increased attention has been devoted to the time differential dual-phase-lag model for the heat conduction: this model relies on good approximation, obtained appropriate expansions of the Taylor series, of the heat equation proposed by Tzou (2015); theory (DPL). As defined in the case of the dual-phase-lag model suggested by Tzou (1995a); the possibility of taking into account higher-order effects in the phase-lags can open very interesting prospects about specific heat transport problems, including the lagging behavior in biological systems. The main impetus in this study is that the interaction among multiple energy carriers progressively gains importance

as the observation scales reduce and have, as a direct result, including high-order terms in the time differential dual-phase-lag heat conduction equation.

In particular, regardless of the number of energy carriers involved, for such biological systems, the ultrafast transient remains governed by the dual-phase-lags but, in parallel, a suitable extension of the original constitutive law toward high-order effects seems to be required: in fact, energy exchanges among the carriers in living tissues will raise the orders of the lagging behavior, adding even more high-order terms in the dual phase-lag heat equation.

Recently, some efforts have been done to change the classical Fourier law of heat conduction by using time- derivative of higher-order in Abouelregal (2019a, 2019b, 2020). In this work, a new generalized thermo-diffusion model with four-phase-lags and higher-order fractional time-derivatives using the approach of fractional calculus has been constructed to describe diffusion and heat conduction in a deformable body. The set of equations that describe solid material deformation when under the simultaneous effect of mechanical fields, temperature and concentration, as well as to introduce a generalized law of the heat and the mass flow are improved. In this model, Fourier's and the Fick's laws were modified to include fractional-order time-derivative of a higher order of the heat flux, the gradient of temperature, the flux of the diffusing mass and the gradient of chemical potential. Special cases of interest are obtained from the current analysis.

In this work a four-phase-lag model is proposed is proposed to incorporate the effect of microstructural interactions on the rapid thermal and diffusion transfer process. In numerical application, the thermo-diffusion response of a half-space medium exposed to a thermal and chemical potential shock is studied. The numerical results temperature, chemical potential, thermal stress and displacement, are obtained. The effects of some physical parameters on the system response are investigated and analyzed. It can be seen that the fractional and the higher orders of the fractional derivative model have significant effects on the dynamic responses of the system. It can be seen that the higher-order of the fractional derivative model has significant effect on the dynamic responses of the system.

## 2. Fractional thermodiffusive model with four-phase-lags

The governing equations for a homogeneous isotropic generalized thermodiffusive solid in the absence of body forces are, as follows (Nowacki 1974a; Sherief, Hamza, and Saleh 2004; Aouadi 2007):

The constitutive equations

$$\sigma_{ij} = 2\mu e_{ij} + \delta_{ij}\left[\lambda e_{ij} - \beta_1\theta - \beta_2 C\right] \tag{1}$$

The strain-displacement relations

$$2e_{ij} = u_{j,i} + u_{i,j} \tag{2}$$

The energy balance equation

$$\rho C_e \frac{\partial\theta}{\partial t} + \beta_1 T_0 \frac{\partial}{\partial t}(\operatorname{div}\boldsymbol{u}) + aT_0\frac{\partial C}{\partial t} = -\operatorname{div}\boldsymbol{q} + Q, \tag{3}$$

The equation of mass conservation

$$\operatorname{div}\ \boldsymbol{\eta} = -\frac{\partial C}{\partial t}, \tag{4}$$

The chemical potential

$$P = -\beta_2 e_{kk} + bC - a\theta \tag{5}$$

The classical Fourier's law

$$\boldsymbol{q}(x,t) = -K\nabla\theta(x,t), \tag{6}$$

The flow of the diffusion molar mass vector

$$\boldsymbol{\eta}(x,t) = -D\nabla P(x,t), \tag{7}$$

Extending the thermoelastic model introduced by Sherief, Hamza, and Saleh (2004), we proposed a modified four-phase-lag thermodiffusive model. In this model, the generalized constitutive equation for heat conduction and diffusion are proposed to describe the lagging behavior are of the form:

$$\mathrm{q}(\boldsymbol{x}, t+\tau_q) = -K\nabla\theta(\boldsymbol{x}, t+\tau_\theta), \tag{8}$$

$$\boldsymbol{\eta}(\boldsymbol{x}, t+\tau_\eta) = -D\nabla P(\boldsymbol{x}, t+\tau_p), \tag{9}$$

Applying Taylor's series of time-fractional order developed in Jumarie (2010), for any positive integer $k$ and any fractional-order $\alpha$ $(0 < \alpha \leq 1)$ to expand all the functions given in (8) and (9) and keeping the terms up to specific orders in $\tau_q$, $\tau_\theta$, $\tau_\eta$ and $\tau_p$ in order to achieve a comparable accuracy

$$\begin{aligned} \boldsymbol{q}(\boldsymbol{x}, t+\tau_q) &= \left(1 + \sum_{k=1}^{l} \frac{(\tau_q)^{k\alpha}}{\Gamma(1+k\alpha)} \frac{\partial^{k\alpha}}{\partial t^{k\alpha}}\right) \boldsymbol{q}(\boldsymbol{x}, t) \\ \nabla\theta(x, t+\tau_\theta) &= \left(1 + \sum_{k=1}^{m} \frac{(\tau_\theta)^{k\alpha}}{\Gamma(1+k\alpha)} \frac{\partial^{k\alpha}}{\partial t^{k\alpha}}\right) \nabla\theta(\boldsymbol{x}, t), \end{aligned} \tag{10}$$

$$\begin{aligned} \boldsymbol{\eta}(\boldsymbol{x}, t+\tau_\eta) &= \left(1 + \sum_{k=1}^{n} \frac{(\tau_\eta)^{k\alpha}}{\Gamma(1+k\alpha)} \frac{\partial^{k\alpha}}{\partial t^{k\alpha}}\right) \boldsymbol{\eta}(\boldsymbol{x}, t), \\ \nabla P(\boldsymbol{x}, t+\tau_p) &= \left(1 + \sum_{k=1}^{h} \frac{(\tau_p)^{k\alpha}}{\Gamma(1+k\alpha)} \frac{\partial^{k\alpha}}{\partial t^{k\alpha}}\right) \nabla P(\boldsymbol{x}, t), \end{aligned} \tag{11}$$

where $\frac{\partial^{k\alpha}}{\partial t^{k\alpha}}(f)$ is the time derivative of order $k\alpha$ of any function $f(t)$ and $\Gamma(1+k\alpha) = (k\alpha)!$. We have expanded until the order $l\alpha$ the heat flux, to the order $m\alpha$ the temperature gradient, to the order $n\alpha$ diffusing mass and to the order $h\alpha$ chemical potential. In the above equation, we have taken into account the Riemann-Liouville fractional integral, which is introduced as a natural generalization of the convolution type integral (Caputo and Mainardi 1971a; Miller and Ross 1993; Podlubny 1999)

$$\frac{\partial^{\alpha}}{\partial t^{\alpha}} f(t) = I^{\alpha} f(t) = \int_0^t \frac{(t-\tau)^{\alpha-1}}{\Gamma(\alpha-1)} f(\tau)\mathrm{d}\tau \tag{12}$$

where $I^{\alpha}$ is the Riemann–Liouville fractional integral operator of order $\alpha$, $f(t)$ is a Lebesgue integrable function and $t$ is the time. In the case that $f(t)$ is absolutely continuous, then

$$\lim_{\alpha\to 1}\left(\frac{\mathrm{d}^{\alpha}}{\mathrm{d}t^{\alpha}} f(t)\right) = f'(t) \tag{13}$$

The integrated differential operator in (12) is of a Caputo-type (Caputo 1974). It has a memory fading due to the convolution with $t^{-\alpha}$. Here, if the parameter$\alpha = 1$ we say that the system has a perfect memory and if $\alpha = 0$ there is no memory (Baker, Eldred, and Palazotto 1996). For any value the parameter $\alpha, (0 < \alpha \leq 1)$ the system has partial memory. Substituting from Eqs. (10) and (11) into Eqs. (6) and (7), we get the fractional-Fourier heat conduction and fractional-Fickian diffusion as

$$\left(1+\sum_{k=1}^{l}\frac{(\tau_q)^{k\alpha}}{\Gamma(1+k\alpha)}\frac{\partial^{k\alpha}}{\partial t^{k\alpha}}\right)\boldsymbol{q}(\boldsymbol{x},t)=-K\left(1+\sum_{k=1}^{m}\frac{(\tau_\theta)^{k\alpha}}{\Gamma(1+k\alpha)}\frac{\partial^{k\alpha}}{\partial t^{k\alpha}}\right)\nabla\theta(\boldsymbol{x},t) \tag{14}$$

$$\left(1+\sum_{k=1}^{n}\frac{(\tau_\eta)^{k\alpha}}{\Gamma(1+k\alpha)}\frac{\partial^{k\alpha}}{\partial t^{k\alpha}}\right)\boldsymbol{\eta}(\boldsymbol{x},t)=-D\left(1+\sum_{k=1}^{h}\frac{(\tau_p)^{k\alpha}}{\Gamma(1+k\alpha)}\frac{\partial^{k\alpha}}{\partial t^{k\alpha}}\right)\nabla P(\boldsymbol{x},t) \tag{15}$$

On taking the divergence of Eq. (14) and using Eq. (3), we get

$$\begin{aligned}\left(1+\sum_{k=1}^{l}\frac{(\tau_q)^{k\alpha}}{\Gamma(1+k\alpha)}\frac{\partial^{k\alpha}}{\partial t^{k\alpha}}\right)\left[\rho C_e\frac{\partial\theta}{\partial t}+\beta_1 T_0\frac{\partial}{\partial t}(\mathrm{div}\boldsymbol{u})+aT_0\frac{\partial C}{\partial t}-Q\right]\\ =K\left(1+\sum_{k=1}^{m}\frac{(\tau_\theta)^{k\alpha}}{\Gamma(1+k\alpha)}\frac{\partial^{k\alpha}}{\partial t^{k\alpha}}\right)\nabla^2\theta\end{aligned} \tag{16}$$

In a similar way, taking the divergence of Eq. (15) and using Eqs. (4) and (5), we obtain

$$\frac{\partial}{\partial t}\left(1+\sum_{k=1}^{n}\frac{(\tau_\eta)^{k\alpha}}{\Gamma(1+k\alpha)}\frac{\partial^{k\alpha}}{\partial t^{k\alpha}}\right)C=D\left(1+\sum_{k=1}^{h}\frac{(\tau_p)^{k\alpha}}{\Gamma(1+k\alpha)}\frac{\partial^{k\alpha}}{\partial t^{k\alpha}}\right)\nabla^2 P(\boldsymbol{x},t) \tag{17}$$

Substituting from Eq. (17) in Eq. (5), we arrive at

$$\begin{aligned}\left(1+\sum_{k=1}^{h}\frac{(\tau_p)^{k\alpha}}{\Gamma(1+k\alpha)}\frac{\partial^{k\alpha}}{\partial t^{k\alpha}}\right)\left(Db\ \nabla^2 C-D\beta_2\nabla^2 e_{kk}-Da\ \nabla^2\theta\ \right)\\ =\frac{\partial}{\partial t}\left(1+\sum_{k=1}^{n}\frac{(\tau_\eta)^{k\alpha}}{\Gamma(1+k\alpha)}\frac{\partial^{k\alpha}}{\partial t^{k\alpha}}\right)C\end{aligned} \tag{18}$$

The equations of the generalized thermo-diffusion comprise the constitutive equations (1) and (5), the heat conduction equation (17), the mass diffusion equation (18) and the following motion equation:

$$\mu u_{i,jj}+(\lambda+\mu)u_{j,ij}-\beta_1\theta_{,i}-\beta_2 C_{,i}+F_i=\rho\ddot{u}_i \tag{19}$$

## 3. Special cases of thermoelasticity and thermo-diffusion theories

In the context of the generalized thermoelastic-diffusion model with four-phase-lag and higher-order time-derivatives (GHTD), we can deduce several special cases. The obtained results are valid for some special cases which can be deduced from our generalized model.

- First, neglecting the effects of diffusion $(a=b=D=\beta_2=0)$ and without fractional-order time derivatives $(\alpha=1)$, we get:
  i. the classical thermoelasticity theory (CTE) (Biot 1956) when $\tau_q=\tau_\theta=0$ and the heat conduction equation associated with this theory will be in the form

$$K\nabla^2\theta=\rho C_e\frac{\partial\theta}{\partial t}+\gamma T_0\frac{\partial e}{\partial t}-Q \tag{20}$$

  ii. Lord-Shulman theory of thermoelasticity (LS) (Lord and Shulman 1967) when $\tau_q=\tau_0>0,\tau_\theta\to 0$, and $l=1$. The heat conduction equation, in this case, takes the form

$$K\nabla^2\theta=\left(1+\tau_0\frac{\partial}{\partial t}\right)\left(\rho C_e\frac{\partial\theta}{\partial t}+\gamma T_0\frac{\partial e}{\partial t}-Q\right) \tag{21}$$

  iii. the generalized heat conduction equation of Tzou theory (DPL) (Tzou 1995b) from (11) when $l=1,\ m=1$

$$K\left(1+\tau_\theta\frac{\partial}{\partial t}\right)\nabla^2\theta=\left(1+\tau_q\frac{\partial}{\partial t}+\frac{\tau_q^2}{2}\frac{\partial^2}{\partial t^2}\right)\left(\rho C_e\frac{\partial\theta}{\partial t}+\gamma T_0\frac{\partial e}{\partial t}-Q\right) \tag{22}$$

iv. the heat equation agrees with Chandrasekharaiah (1998); (CT) when $l=1$ and $m=1$

$$K\left(1+\tau_\theta\frac{\partial}{\partial t}\right)\nabla^2\theta=\left(1+\tau_q\frac{\partial}{\partial t}\right)\left(\rho C_e\frac{\partial\theta}{\partial t}+\gamma T_0\frac{\partial e}{\partial t}-Q\right) \tag{23}$$

v. the generalized theory with two-phase-lags of high-order time derivatives (HDPL) by taking $\tau_q,\ \tau_\theta>0,\ l>2,\ m>1$.

- Second, in case of the thermo-diffusion interactions is taken into account and neglecting the fractional-order time-derivatives $(\alpha=1)$, we obtain:

i- the classical theory of thermodiffusion developed by Nowacki (1974a, 1974b) (CTD), when $a,b,D,\beta_2>0$ and the phase-lags vanish i.e. $\tau_q=\ \tau_\theta=\tau_\eta=\tau_p=0$. The basic equations will be in the forms

$$\rho C_e\frac{\partial\theta}{\partial t}+\beta_1 T_0\frac{\partial}{\partial t}(\text{div }\boldsymbol{u})+aT_0\frac{\partial C}{\partial t}-Q=K\nabla^2\theta \tag{24}$$

$$\frac{\partial C}{\partial t}=Db\ \nabla^2 C-D\beta_2\nabla^2 e_{kk}-Da\ \nabla^2\theta \tag{25}$$

ii- the generalized theory of thermo-diffusion introduced by Sherief, Hamza, and Saleh (2004); (STD) by putting $l=1,\ k=1$ and $\tau_\theta=\tau_p=0$. The the heat conduction and the mass diffusion equations are given by:

$$\left(1+\tau_q\frac{\partial}{\partial t}\right)\left(\rho C_e\frac{\partial\theta}{\partial t}+\beta_1 T_0\frac{\partial}{\partial t}(\text{div }\boldsymbol{u})+aT_0\frac{\partial C}{\partial t}-Q\right)=K\nabla^2\theta \tag{26}$$

$$\left(1+\tau_\eta\frac{\partial}{\partial t}\right)\frac{\partial C}{\partial t}=Db\ \nabla^2 C-D\beta_2\nabla^2 e_{kk}-Da\ \nabla^2\theta \tag{27}$$

iii- the generalized thermo-diffusion theory with four-phase-lags of high-order time derivatives (HTPL) by taking$\tau_q,\ \tau_\theta,\tau_\eta,\tau_p>0,\ l,m,n,h\geq 1$.

- Third, taken into account the thermo-diffusion interactions and fractional order time derivatives $(0<\alpha\leq 1)$, we obtain:

i. The fractional thermo-diffusion theory introduced by Ezzat and Fayik (2011); (FTD) by putting $l=2,\ m=n=p=1,\ $ and $\tau_p=0$. The the heat conduction and the mass diffusion equations are given by:

$$\left(1+\frac{\tau_q^\alpha}{\alpha!}\frac{\partial^\alpha}{\partial t^\alpha}+\frac{\tau_q^{2\alpha}}{(2\alpha)!}\frac{\partial^{2\alpha}}{\partial t^{2\alpha}}\right)\left(\rho C_e\frac{\partial\theta}{\partial t}+\beta_1 T_0\frac{\partial}{\partial t}(div\ \boldsymbol{u})+aT_0\frac{\partial C}{\partial t}-Q\right)=K\left(1+\frac{\tau_\theta^\alpha}{\alpha!}\frac{\partial^\alpha}{\partial t^\alpha}\right)\nabla^2\theta \tag{28}$$

$$\left(1+\frac{\tau_\eta^\alpha}{\alpha!}\frac{\partial^\alpha}{\partial t^\alpha}\right)\frac{\partial C}{\partial t}=Db\ \nabla^2 C-D\beta_2\nabla^2 e_{kk}-Da\ \nabla^2\theta \tag{29}$$

ii The generalized fractional thermo-diffusion model with four-phase-lags of high-order time derivatives (HFTD) by taking $\tau_q,\ \tau_\theta,\tau_\eta,\tau_p>0,\ l,m,n,h\geq 1$.

## 4. Application of the modified thermo-diffusion model

In this section, the problem of an isotropic thermoelastic half-space $(x\geq 0)$, based on the generalized fractional thermo-diffusion model with four-phase-lags and high-order time derivatives

(HFTD), has been studied. The plane $x = 0$ of the half-space is free of traction and is exposed to a thermal shock. Chemical potential is also assumed to be a known function of time on the same plane. It is assumed that all studied functions are bounded and vanish as $x \to \infty$. It follows from the description of the problem that all considered functions will depend on $x$ and $t$ only. We thus, obtain the displacement components of the form,

$$u_x = u(x,t), \;\; u_y = u_z = 0 \tag{30}$$

The strain component is given by

$$e_{xx} = e = \frac{\partial u}{\partial x} \tag{31}$$

From Eqs. (1) and (5), we get

$$\begin{aligned} \sigma_{xx} &= (\lambda + 2\mu)\frac{\partial u}{\partial x} - \beta_1\theta - \beta_2 C \\ \sigma_{yy} &= \lambda\frac{\partial u}{\partial x} - \beta_1\theta - \beta_2 C \end{aligned} \tag{32}$$

$$P = -\beta_2\frac{\partial u}{\partial x} + bC - a\theta \tag{33}$$

Equations (16), (18), and (19), are reduce to

$$(\lambda + \mu)\frac{\partial e}{\partial x} + \mu\frac{\partial^2 u}{\partial x^2} - \beta_1\frac{\partial \theta}{\partial x} - \beta_2\frac{\partial C}{\partial x} = \rho\frac{\partial^2 u}{\partial t^2} \tag{34}$$

$$\begin{aligned} \left(1 + \sum_{k=1}^{l}\frac{(\tau_q)^{k\alpha}}{\Gamma(1+k\alpha)}\frac{\partial^{k\alpha}}{\partial t^{k\alpha}}\right)\left[\rho C_e\frac{\partial \theta}{\partial t} + \beta_1 T_0\frac{\partial}{\partial t}(\text{div } \boldsymbol{u}) + aT_0\frac{\partial C}{\partial t} - Q\right] \\ = K\left(1 + \sum_{k=1}^{m}\frac{(\tau_\theta)^{k\alpha}}{\Gamma(1+k\alpha)}\frac{\partial^{k\alpha}}{\partial t^{k\alpha}}\right)\frac{\partial^2 \theta}{\partial x^2} \end{aligned} \tag{35}$$

$$\begin{aligned} \left(1 + \sum_{k=1}^{h}\frac{(\tau_p)^{k\alpha}}{\Gamma(1+k\alpha)}\frac{\partial^{k\alpha}}{\partial t^{k\alpha}}\right)\left(Db\;\frac{\partial^2 C}{\partial x^2} - D\beta_2\nabla^2\frac{\partial^2 e}{\partial x^2} - Da\frac{\partial^2 \theta}{\partial x^2}\right) \\ = \frac{\partial}{\partial t}\left(1 + \sum_{k=1}^{n}\frac{(\tau_\eta)^{k\alpha}}{\Gamma(1+k\alpha)}\frac{\partial^{k\alpha}}{\partial t^{k\alpha}}\right)C \end{aligned} \tag{36}$$

To solve the problem, we will use the following dimensionless variables:

$$\begin{aligned} \{x',\;\; u'\} = c_0\eta\{x,u\},\;\; \{t',\;\; \tau'_q, \tau'_\theta, \tau'_\eta, \tau'_\vartheta\} = c_0^2\eta\{t,\;\; \tau_q, \tau_\theta, \tau_\eta, \tau_\vartheta\}, P' = \frac{P}{K}, \\ \theta = \frac{\beta_1\theta}{(\lambda+2\mu)}, C' = \frac{\beta_2 C}{(\lambda+2\mu)},\;\; \sigma'_{ij} = \frac{\sigma_{ij,}}{(\lambda+2\mu)},\;\; \eta = \frac{\rho C_E}{K},\;\; c_0 = \sqrt{\frac{\lambda+2\mu}{\rho}} \end{aligned} \tag{37}$$

In view of Eq. (37), the non-dimensional form of Eqs. (32)–(36) reduce to

$$\frac{\partial^2 u}{\partial x^2} - \frac{\partial \theta}{\partial x} - \frac{\partial C}{\partial x} = \frac{\partial^2 u}{\partial t^2} \tag{38}$$

$$\left(1 + \sum_{k=1}^{l}\frac{(\tau_q)^{k\alpha}}{\Gamma(1+k\alpha)}\frac{\partial^{k\alpha}}{\partial t^{k\alpha}}\right)\left[\frac{\partial \theta}{\partial t} + \varepsilon\frac{\partial e}{\partial t} + \alpha_1\varepsilon\frac{\partial C}{\partial t}\right] = \left(1 + \sum_{k=1}^{m}\frac{(\tau_\theta)^{k\alpha}}{\Gamma(1+k\alpha)}\frac{\partial^{k\alpha}}{\partial t^{k\alpha}}\right)\frac{\partial^2 \theta}{\partial x^2} \tag{39}$$

$$\left(1+\sum_{k=1}^{h}\frac{(\tau_p)^{k\alpha}}{\Gamma(1+k\alpha)}\frac{\partial^{k\alpha}}{\partial t^{k\alpha}}\right)\left(\alpha_3\frac{\partial^2 C}{\partial x^2}-\frac{\partial^2 e}{\partial x^2}-\alpha_1\frac{\partial^2\theta}{\partial x^2}\right)=\alpha_2\frac{\partial}{\partial t}\left(1+\sum_{k=1}^{n}\frac{(\tau_\eta)^{k\alpha}}{\Gamma(1+k\alpha)}\frac{\partial^{k\alpha}}{\partial t^{k\alpha}}\right)C \tag{40}$$

$$\begin{aligned}\sigma_{xx}&=e-\theta-C\\ \sigma_{yy}&=\left(1-2/\beta^2\right)e-\theta-C\end{aligned} \tag{41}$$

$$P=\alpha_3 C-e-\alpha_1\theta \tag{42}$$

where

$$\varepsilon=\frac{T_0\beta_1^2}{\rho C_e(\lambda+2\mu)},\ \alpha_1=\frac{a(\lambda+2\mu)}{\beta_1\beta_2},\ \alpha_2=\frac{(\lambda+2\mu)}{\eta D\beta_2^2},\ \alpha_3=\frac{b(\lambda+2\mu)}{\beta_2^2},\ \beta^2=\frac{\lambda+2\mu}{\mu}. \tag{43}$$

## 5. Initial and boundary conditions

The initial conditions are

$$\begin{aligned}&u(x,0)=\frac{\partial^k u(x,0)}{\partial t^k}=0,\quad \theta(x,0)=\frac{\partial^k\theta(x,0)}{\partial t^k}=0, P(x,0)=\frac{\partial^k P(x,0)}{\partial t^k}=0,\\ &C(x,0)=\frac{\partial^k C(x,0)}{\partial t^k}=0,\quad k=1,2,3,\ldots,(l-1,m-1,h-1,n-1)\end{aligned} \tag{44}$$

The homogeneous initial conditions are supplemented by the following boundary conditions:

$$\sigma_{xx}(x,0)=0,\ \ \theta(x,h)=\theta_1 H(t), P(x,h)=P_1 H(t) \tag{45}$$

where $\theta_1$ and $P_1$ are constants.

## 6. The solution in the transformed domain

Using Laplace transform technique, Eqs. (38)–(42) can be transformed into the form

$$\frac{d^2\overline{u}}{dx^2}-s^2\overline{u}=\frac{d\overline{\theta}}{dx}+\frac{d\overline{C}}{dx} \tag{46}$$

$$s\left(1+\sum_{k=1}^{l}\frac{(\tau_q)^{k\alpha}}{\Gamma(1+k\alpha)}s^{k\alpha}\right)\left[\overline{\theta}+\varepsilon\overline{e}+\alpha_1\varepsilon\overline{C}\right]=\left(1+\sum_{k=1}^{m}\frac{(\tau_\theta)^{k\alpha}}{\Gamma(1+k\alpha)}s^{k\alpha}\right)\frac{d^2\overline{\theta}}{dx^2} \tag{47}$$

$$\left(1+\sum_{k=1}^{h}\frac{(\tau_p)^{k\alpha}}{\Gamma(1+k\alpha)}s^{k\alpha}\right)\left(\alpha_3\frac{d^2\overline{C}}{dx^2}-\frac{d^2\overline{e}}{dx^2}-\alpha_1\frac{d^2\overline{\theta}}{dx^2}\right)=s\alpha_2\left(1+\sum_{k=1}^{n}\frac{(\tau_\eta)^{k\alpha}}{\Gamma(1+k\alpha)}s^{k\alpha}\right)\overline{C} \tag{48}$$

$$\begin{aligned}\overline{\sigma}_{xx}&=\overline{e}-\overline{\theta}-\overline{C}\\ \overline{\sigma}_{yy}&=\left(1-2/\beta^2\right)\overline{e}-\overline{\theta}-\overline{C}\end{aligned} \tag{49}$$

$$\overline{P}=\alpha_3\overline{C}-\overline{e}-\alpha_1\overline{\theta} \tag{50}$$

Eliminating $\overline{e}$ and $\overline{C}$ among Eqs. (46)–(48), we obtain

$$\left(\frac{d^2}{dx^2}-m_1^2\right)\left(\frac{d^2}{dx^2}-m_2^2\right)\left(\frac{d^2}{dx^2}-m_3^2\right)\overline{\theta}=0 \tag{51}$$

where $m_1^2, m_2^2$ and $m_3^2$ are the roots of the characteristic equation

$$m^6 - a_1 m^4 + a_2 m^2 - a_3 = 0 \tag{52}$$

where

$$\begin{aligned}
a_1 &= \frac{\mathrm{s}}{\alpha_3 - 1}\left[\alpha_4\alpha_1\varepsilon(\alpha_1+2) + \alpha_4\alpha_3(1+\varepsilon) - \alpha_4 + \alpha_2\alpha_5 + \alpha_3 s\right],\\
a_2 &= \frac{\mathrm{s}^2}{\alpha_3 - 1}\left[\alpha_4\varepsilon s\alpha_1^2 + \alpha_4\alpha_3 s + \alpha_4\alpha_5\alpha_2(1+\varepsilon) + \alpha_2 s\alpha_5\right],\\
a_2 &= \frac{\mathrm{s}^4\alpha_4\alpha_5\alpha_2}{\alpha_3 - 1}, \quad \alpha_4 = \frac{1 + \sum_{k=1}^{l}\frac{(\tau_q)^{k\alpha}}{\Gamma(1+k\alpha)}s^{k\alpha}}{1 + \sum_{k=1}^{m}\frac{(\tau_\theta)^{k\alpha}}{\Gamma(1+k\alpha)}s^{k\alpha}}, \quad \alpha_5 = \frac{1 + \sum_{k=1}^{n}\frac{(\tau_\eta)^{k\alpha}}{\Gamma(1+k\alpha)}s^{k\alpha}}{1 + \sum_{k=1}^{h}\frac{(\tau_p)^{k\alpha}}{\Gamma(1+k\alpha)}s^{k\alpha}}.
\end{aligned} \tag{53}$$

In a similar manner we can show that $\overline{e}$ and $\overline{C}$ satisfy the equations

$$\left(\frac{d^2}{dx^2} - m_1^2\right)\left(\frac{d^2}{dx^2} - m_2^2\right)\left(\frac{d^2}{dx^2} - m_3^2\right)\{\overline{e}, \overline{C}\} = 0 \tag{54}$$

The solution of Eqs. (51) and (54) has the form,

$$\overline{\theta} = \sum_{n=1}^{3} A_n \mathrm{e}^{-m_n x} \tag{55}$$

$$\overline{e} = \sum_{n=1}^{3} A'_n \mathrm{e}^{-m_n x} \tag{56}$$

$$\overline{C} = \sum_{n=1}^{3} A''_n \mathrm{e}^{-m_n x} \tag{57}$$

where $A_n$, $A'_n$, and $A''_n$ are parameters depending only on the parameters.Substituting from Eqs. (55)–(57) into Eqs. (46)–(48), we get

$$\begin{aligned}
A'_n &= \frac{m_n^2\left[m_n^2 - s\alpha_4(1-\alpha_1\varepsilon)\right]}{s\alpha_4\varepsilon\left[m_n^2(1+\alpha_1) - \alpha_1\mathrm{s}^2\right]}A_n = \Omega_n A_n\\
A''_n &= \frac{m_n^4 - m_n^2\left[\mathrm{s}^2 + s\alpha_4(1+\varepsilon)\right] + \alpha_4\mathrm{s}^4}{s\alpha_4\varepsilon\left[m_n^2(1+\alpha_1) - \alpha_1\mathrm{s}^2\right]}A_n = \Gamma_n A_n
\end{aligned} \tag{58}$$

We thus have

$$\overline{e} = \sum_{n=1}^{3} \Omega_n A_n \mathrm{e}^{-m_n x} \tag{59}$$

$$\overline{C} = \sum_{n=1}^{3} \Gamma_n A_n \mathrm{e}^{-m_n x} \tag{60}$$

Integrating both sides of Eq. (31), we obtain upon using the relation, Eq. (55),

$$\overline{u} = -\sum_{n=1}^{3} \frac{\Omega_n}{m_n} A_n \mathrm{e}^{-m_n x} \tag{61}$$

Substituting from Eqs. (55, 60), and (61) into Eqs. (49) and (50), we get

$$\overline{\sigma}_{xx} = \sum_{n=1}^{3} (\Omega_n - \Gamma_n - 1) A_n \mathrm{e}^{-m_n x} \tag{62}$$

$$\overline{P} = \sum_{n=1}^{3} (\alpha_3\Gamma_n - \Omega_n - \alpha_1) A_n \mathrm{e}^{-m_n x} \tag{63}$$

In order to evaluate the unknown parameters $A_n$, $(n = 1, 2, 3)$, we will use the Laplace transform of the boundary conditions (41), together with Eqs. (55, 62) and (63). We, thus, arrive at the following set of linear equations:

$$\sum_{n=1}^{3} A_n = \theta_1/s = \bar{f}_1(s) \tag{64}$$

$$\sum_{n=1}^{3} (\Omega_n - \Gamma_n - 1)A_n = 0 \tag{65}$$

$$\sum_{n=1}^{3} (\alpha_3\Gamma_n - \Omega_n - \alpha_1)A_n = P_1/s = \bar{f}_2(s) \tag{66}$$

In order to determine the studied fields in the physical domain, a numerical inversion method is used to obtain the numerical results. The details of these methods can be found in Durbin (1974).

## 7. Results and discussion

In this section, the numerical solution is carried out for copper material to illustrate the efficiency and accuracy of the introduced new model (HFTD) and compare theoretical results. The numerical analysis was performed using the procedure proposed by Durbin (1974), with the help of ***MATHEMATICA*** programing.

We have made discussions to highlight the effects of higher-order fractional time-derivatives and the four phase-lag parameters $\tau_q, \tau_\theta, \tau_\eta, \tau_p$ on all studied physical fields. According to Zenkour, Mashat, and Abouelregal (2012) and Zenkour, Alzahrani, and Abouelregal (2015), the following physical parameters are given

$$C_E = 383.1\frac{\mathrm{J}}{(\mathrm{kgK})},\ T_0 = 296\ (\mathrm{K}),\ \alpha_t = 1.78\times10^{-5}\left(\frac{1}{\mathrm{K}}\right),\ K = 386\left(\frac{\mathrm{W}}{\mathrm{m\ K}}\right),$$

$$\lambda = 7.76\times10^{10}\left(\frac{\mathrm{N}}{\mathrm{m}^2}\right),\ \mu = 3.86\times10^{10}\left(\frac{\mathrm{N}}{\mathrm{m}^2}\right),\ \alpha_c = 1.98\times10^{-4}\left(\frac{\mathrm{m}^3}{\mathrm{kg}}\right),\ \rho = 8954\ \left(\frac{\mathrm{kg}}{\mathrm{m}^3}\right),$$

$$t = 0.05\mathrm{s},\ D = 0.85\times10^{-8}\left(\frac{\mathrm{kg\ s}}{\mathrm{m}^3}\right),\ a = 1.2\times10^{4}\left(\frac{\mathrm{m}^2}{\mathrm{K\ s}^2}\right),\ b = 0.9\times10^{6}\left(\frac{\mathrm{m}^5}{\mathrm{kg\ s}^2}\right).$$

The non-dimensional distributions of displacement $u$, temperature $\theta$, and stress $\sigma_{xx}$, concentration $C$ and chemical potential $P$ fields have been evaluated versus the $x$-axis in the range $0 \le x \le 5$. The results are performed for the values of phase-lags, namely $\tau_q = 0.1$, $\tau_\theta = 0.05$, $\tau_\eta = 0.1$ and $\tau_p = 0.05$. Other physical parameters are fixed.

### 7.1. Comparison between different models of thermo-diffusion

In this case, the results of the models of thermo-diffusion with thermal and diffusion relaxation times (STD), fractional thermo-diffusion (FTD) are compared with the generalized thermo-diffusion model with higher-order fractional time-derivatives and four phase-lags (HFTD).The comparison results are displayed in Tables 1–5. From Tables 1–5 it is concluded that:

- The classical and generalized thermoelasticity theories can be deduced from our results by neglecting the diffusion effect.
- The higher-order fractional time-derivatives model has a great effect on physical fields.
- The system response depends on the values of the higher-order parameters $l, m, n$ and $h$.
- The values of temperature $\theta$, and the thermal stress $\sigma_{xx}$ are highlysensitive to the variation of the parameters $l, m, n$ and $h$. However, the concentration $C$ and chemical potential $P$ are less sensitive to the variation of the higher fractional order parameters.
- The axial displacement $u$ is not affected by the parameters $l, m, n$ and $h$.

**Table 1.** Effect of the higher-order fractional time-derivatives on the temperature $\theta$.

| | | | HFTD ($\alpha = 0.7,\ \tau_q = 0.02,\ \tau_\theta = 0.01,\ \tau_\eta = 0.02$ and $\tau_p = 0.01$) | | | | | |
|---|---|---|---|---|---|---|---|---|
| $x$ | STD ($\alpha = 1$) | FTD ($\alpha = 0.7$) | $l=1, m=1, n=1, h=1$ | $l=2, m=1, n=2, h=1$ | $l=2, m=2, n=2, h=2$ | $l=3, m=2, n=2, h=2$ | $l=4, m=2, n=2, h=2$ | $l=5, m=2, n=2, h=2$ |
| 0 | 1.01666 | 1.01666 | 1.01666 | 1.01666 | 1.01666 | 1.01666 | 1.01666 | 1.01666 |
| 0.5 | 0.483008 | 0.688794 | 0.658576 | 0.68876 | 0.660537 | 0.722581 | 0.909078 | 1.05687 |
| 1 | 0.3765 | 0.344044 | 0.373842 | 0.344002 | 0.347035 | 0.282308 | 0.418437 | 0.558496 |
| 1.5 | 0.182262 | 0.227506 | 0.223432 | 0.227475 | 0.223573 | 0.26626 | 0.348218 | 0.0540859 |
| 2 | 0.0855901 | 0.122204 | 0.136036 | 0.122084 | 0.127236 | 0.12426 | 0.190086 | 0.192037 |
| 2.5 | 0.0632791 | 0.0753746 | 0.0806497 | 0.0752961 | 0.0751199 | 0.148116 | 0.00180933 | 0.180298 |
| 3 | 0.0294155 | 0.0429942 | 0.0470162 | 0.0428702 | 0.043236 | 0.000278898 | −0.00789305 | −0.217278 |
| 3.5 | 0.0203023 | 0.0234126 | 0.0267637 | 0.0232637 | 0.0238263 | 0.0666318 | −0.19804 | −0.112001 |
| 4 | 0.00831289 | 0.0125057 | 0.0145021 | 0.012324 | 0.0125009 | 0.0116268 | −0.0120029 | 0.181871 |
| 4.5 | 0.00239196 | 0.00549067 | 0.0071086 | 0.0053511 | 0.0056805 | 0.0146613 | −0.240486 | −0.0843079 |
| 5 | 0.00040123 | 0.00168658 | 0.0026894 | 0.0016439 | 0.0017416 | −0.0150232 | 0.0806953 | −0.200142 |

**Table 2.** Effect of the higher-order fractional time-derivatives on the displacement $u$.

| | | | HFTD ($\alpha = 0.7,\ \tau_q = 0.02,\ \tau_\theta = 0.01,\ \tau_\eta = 0.02$ and $\tau_p = 0.01$) | | | | | |
|---|---|---|---|---|---|---|---|---|
| $x$ | STD ($\alpha = 1$) | FTD ($\alpha = 0.7$) | $l=1, m=1, n=1, h=1$ | $l=2, m=1, n=2, h=1$ | $l=2, m=2, n=2, h=2$ | $l=3, m=2, n=2, h=2$ | $l=4, m=2, n=2, h=2$ | $l=5, m=2, n=2, h=2$ |
| 0 | −0.160633 | −0.165169 | −0.166819 | −0.165193 | −0.165364 | −0.164782 | −0.164801 | −0.166866 |
| 0.5 | 0.0200813 | 0.0202494 | 0.0200968 | 0.0202325 | 0.0202016 | 0.0203687 | 0.0242045 | 0.0238996 |
| 1 | 0.0104723 | 0.0113498 | 0.011647 | 0.0113392 | 0.01137 | 0.0112588 | 0.00983406 | 0.0100316 |
| 1.5 | 0.00350411 | 0.00401061 | 0.0042001 | 0.0040036 | 0.0040248 | 0.00391712 | 0.000255396 | 0.00106932 |
| 2 | 0.00132418 | 0.00151617 | 0.0015951 | 0.0015113 | 0.0015207 | 0.0014476 | 0.00225991 | 0.00340418 |
| 2.5 | 0.000684816 | 0.000733992 | 0.0007603 | 0.0007322 | 0.0007347 | 0.000712549 | 0.000272587 | 0.0033958 |
| 3 | 0.000466331 | 0.000467119 | 0.0004741 | 0.0004682 | 0.0004679 | 0.000477512 | −0.00312306 | −0.0018656 |
| 3.5 | 0.000364146 | 0.00035449 | 0.0003552 | 0.0003566 | 0.0003562 | 0.000375348 | 0.00119638 | −0.00214871 |
| 4 | 0.000298219 | 0.000290755 | 0.00029 | 0.0002932 | 0.0002931 | 0.000295547 | 0.00292144 | 0.00284395 |
| 4.5 | 0.000252062 | 0.000244976 | 0.0002447 | 0.0002478 | 0.0002475 | 0.000243285 | −0.00248102 | 0.00242904 |
| 5 | 0.0002141 | 0.000208349 | 0.0002089 | 0.0002111 | 0.0002108 | 0.000212441 | −0.00193829 | −0.00257289 |

**Table 3.** Effect of the higher-order fractional time-derivatives on the stress component $\sigma_{xx}$.

| | | | HFTD ($\alpha = 0.7,\ \tau_q = 0.02,\ \tau_\theta = 0.01,\ \tau_\eta = 0.02$ and $\tau_p = 0.01$) | | | | | |
|---|---|---|---|---|---|---|---|---|
| $x$ | STD ($\alpha = 1$) | FTD ($\alpha = 0.7$) | $l=1, m=1, n=1, h=1$ | $l=2, m=1, n=2, h=1$ | $l=2, m=2, n=2, h=2$ | $l=3, m=2, n=2, h=2$ | $l=4, m=2, n=2, h=2$ | $l=5, m=2, n=2, h=2$ |
| 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 0.5 | −0.307369 | −0.381298 | −0.380125 | −0.381544 | −0.379549 | −0.385719 | −2.33989 | 1.05558 |
| 1 | −0.166167 | −0.174535 | −0.179167 | −0.175348 | −0.175207 | −0.264581 | −2.28734 | 0.561984 |
| 1.5 | −0.0720852 | −0.0866936 | −0.0886646 | −0.0882365 | −0.0878082 | −0.108448 | 0.958086 | −1.20512 |
| 2 | −0.0480317 | −0.0558966 | −0.0562707 | −0.0561707 | −0.0560762 | −0.0358243 | 0.28599 | −1.51182 |
| 2.5 | −0.0377914 | −0.0423771 | −0.0429629 | −0.0308797 | −0.0391201 | −0.0264149 | −2.32049 | 0.538868 |
| 3 | −0.0331934 | −0.0303726 | −0.035643 | −0.0283848 | −0.0272135 | −0.0318254 | 0.0587533 | 1.10097 |
| 3.5 | −0.0311472 | −0.0163017 | −0.0304041 | −0.0246798 | −0.0213008 | −0.0207413 | 2.22497 | −0.809631 |
| 4 | −0.0166329 | −0.0224078 | −0.0260878 | −0.0220142 | −0.0224216 | −0.0187396 | −0.464393 | −1.02997 |
| 4.5 | −0.0154422 | −0.0169709 | −0.0222074 | −0.0201956 | −0.0226952 | −0.0261752 | −1.29413 | 1.01294 |
| 5 | −0.014438 | −0.0158605 | −0.0184169 | −0.0190885 | −0.0170626 | −0.0189758 | 0.538063 | 0.997451 |

- A significant phenomenon observed in all tables is that the values of the physical fields in the fractional thermo-diffusion model with four phase-lags (HFTD) is restricted in a specific region in the medium.
- The distributions according to the HFTD model show the behavior of limited wave propagation speeds based on the values of the parameters $l$, $m$, $n$ and $h$. This differs from the cases in the coupled theories of thermo-diffusion (Nowacki 1974a) where an infinite propagation velocity is inherent.
- It is necessary to take into account the values of the parameters $l, m, n$ and $h$ until the materials attain the steady state.

**Table 4.** Effect of the higher-order fractional time-derivatives on the chemical potential $P$.

| | | | HFTD ($\alpha = 0.7,\ \tau_q = 0.02,\ \tau_\theta = 0.01,\ \tau_\eta = 0.02$ and $\tau_p = 0.01$) | | | | | |
|---|---|---|---|---|---|---|---|---|
| $x$ | STD ($\alpha = 1$) | FTD ($\alpha = 0.7$) | $l=1,\ m=1,$ $n=1,\ h=1$ | $l=2,\ m=1,$ $n=2,\ h=1$ | $l=2,\ m=2,$ $n=2,\ h=2$ | $l=3,\ m=2,$ $n=2,\ h=2$ | $l=4,\ m=2,$ $n=2,\ h=2$ | $l=5,\ m=2,$ $n=2,\ h=2$ |
| 0 | 0.101666 | 0.101666 | 0.101666 | 0.101666 | 0.101666 | 0.101666 | 0.101666 | 0.101666 |
| 0.5 | 0.261691 | 0.241966 | 0.236846 | 0.245275 | 0.245678 | 0.246662 | 7.70911 | −2.77316 |
| 1 | 0.16452 | 0.186283 | 0.18291 | 0.187469 | 0.186664 | 0.188407 | −1.16762 | 6.21046 |
| 1.5 | 0.132041 | 0.11951 | 0.140077 | 0.111451 | 0.106728 | 0.107034 | −1.17243 | −4.68767 |
| 2 | 0.0672146 | 0.0901275 | 0.104984 | 0.0885281 | 0.0901143 | 0.0897325 | 1.92714 | 4.48618 |
| 2.5 | 0.0582288 | 0.0639212 | 0.0743505 | 0.0769196 | 0.0687885 | 0.0685514 | −2.85792 | −4.44156 |
| 3 | 0.0609694 | 0.0562802 | 0.0431804 | 0.0671032 | 0.0697978 | 0.0707653 | 9.80381 | 4.19641 |
| 3.5 | 0.0195722 | 0.0467504 | 0.0181089 | 0.0086251 | 0.0276757 | 0.0280527 | −3.7832 | −3.49128 |
| 4 | 0.0246777 | 0.0287724 | 0.0079564 | 0.0275027 | 0.0110876 | 0.0109457 | 1.52292 | 2.99764 |
| 4.5 | 0.031017 | −0.00643396 | 0.0126899 | 0.0272001 | 0.0286053 | 0.0282844 | −4.80221 | −2.4006 |
| 5 | 0.00431202 | 0.0176697 | 0.0273028 | 0.031248 | 0.0204251 | 0.0203549 | −8.27058 | 1.81951 |

**Table 5.** Effect of the higher-order fractional time-derivatives on the concentration $C$.

| | | | HFTD ($\alpha = 0.7,\ \tau_q = 0.02,\ \tau_\theta = 0.01,\ \tau_\eta = 0.02$ and $\tau_p = 0.01$) | | | | | |
|---|---|---|---|---|---|---|---|---|
| $x$ | STD ($\alpha = 1$) | FTD ($\alpha = 0.7$) | $l=1,\ m=1,$ $n=1,\ h=1$ | $l=2,\ m=1,$ $n=2,\ h=1$ | $l=2,\ m=2,$ $n=2,\ h=2$ | $l=3,\ m=2,$ $n=2,\ h=2$ | $l=4,\ m=2,$ $n=2,\ h=2$ | $l=5,\ m=2,$ $n=2,\ h=2$ |
| 0 | 0.214352 | 0.214352 | 0.214352 | 0.214352 | 0.214352 | 0.214352 | 0.214352 | 0.214352 |
| 0.5 | 0.110095 | 0.115655 | 0.114608 | 0.115938 | 0.115624 | 0.116889 | 2.23604 | −1.25184 |
| 1 | 0.0810905 | 0.0774626 | 0.0768407 | 0.0783641 | 0.0784504 | 0.0895125 | 2.12303 | −0.738089 |
| 1.5 | 0.0538629 | 0.0609018 | 0.0601748 | 0.0626049 | 0.0619847 | 0.0646832 | −1.179 | 1.09486 |
| 2 | 0.0453958 | 0.0515829 | 0.0507981 | 0.0519037 | 0.0516958 | 0.0497063 | −0.308494 | 1.68291 |
| 2.5 | 0.0397046 | 0.0437712 | 0.0439683 | 0.0311687 | 0.0401759 | 0.0390274 | 2.24939 | −0.40179 |
| 3 | 0.0359709 | 0.0325883 | 0.0382076 | 0.0303958 | 0.0291096 | 0.0297093 | −0.287375 | −1.26473 |
| 3.5 | 0.0340092 | 0.0176995 | 0.0331205 | 0.0268749 | 0.02316 | 0.0228619 | −2.27096 | 0.702067 |
| 4 | 0.0182819 | 0.0245326 | 0.0285775 | 0.0240977 | 0.0245314 | 0.024012 | 0.524922 | 1.20749 |
| 4.5 | 0.016965 | 0.0186062 | 0.0243762 | 0.0221406 | 0.0248879 | 0.0254652 | 1.11217 | −0.935062 |
| 5 | 0.0158445 | 0.0173964 | 0.0202326 | 0.0209362 | 0.0187211 | 0.0188876 | −0.751826 | −1.19302 |

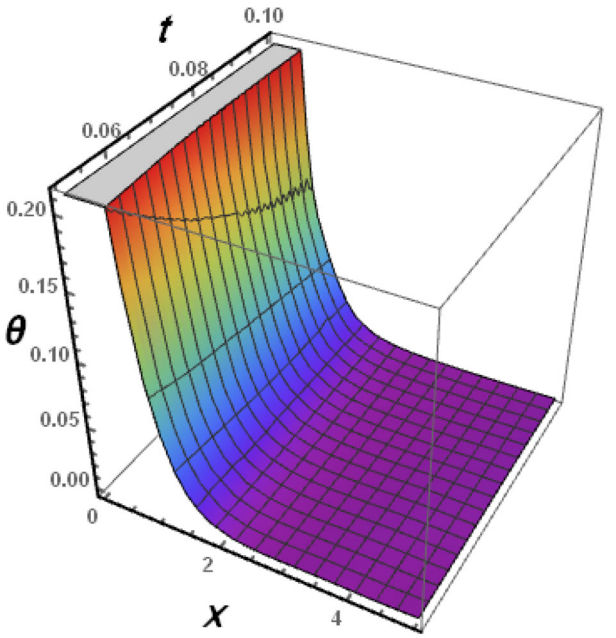


**Figure 1.** The temperature $\theta$ for different times instant $t$ and distance $x$.

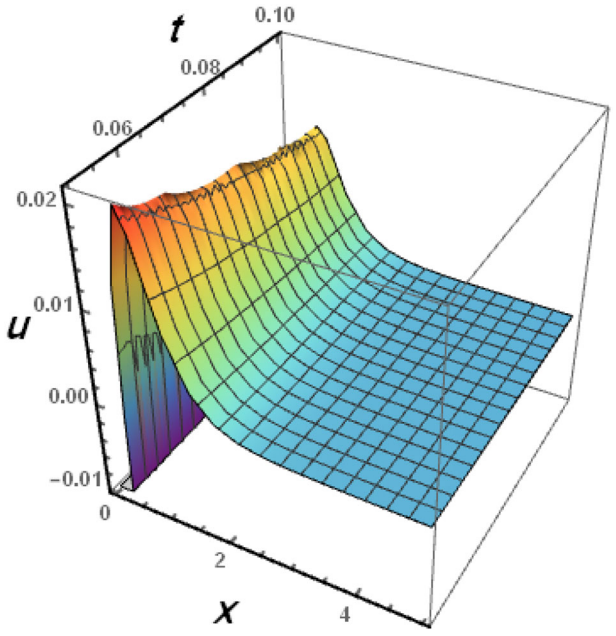


**Figure 2.** The displacement $u$ for different time instant $t$ and distance $x$.

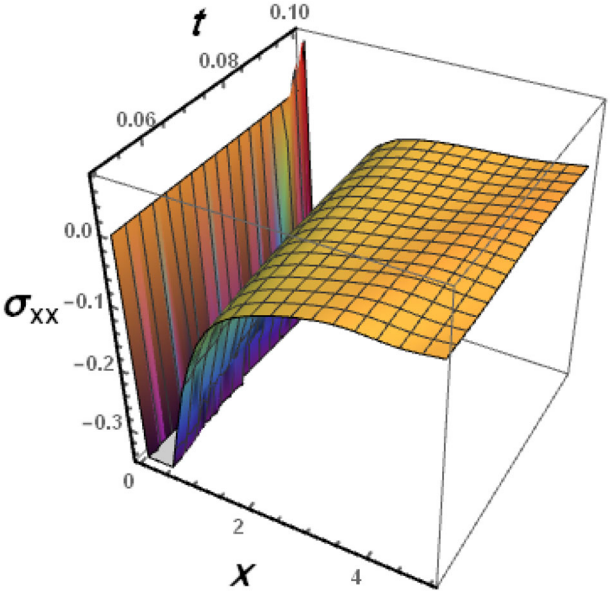


**Figure 3.** The stress $\sigma_{xx}$ for different time instant $t$ and distance $x$.

- It is adequate to take $l = 3$, $m = 2$, $n = 2$ and $h = 2$ to acquire very close and accurate results.
- As a very important observation, in the case of HFTD theory, with increasing the parameters $l, m, n$ and $h$, yields accurate results. This corresponds to the results obtained by Abouelregal (2019a, 2019b, 2020).

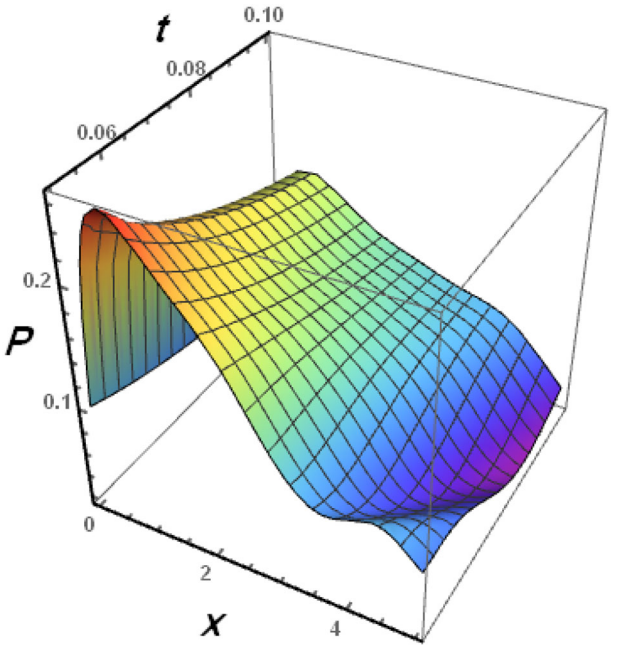


**Figure 4.** The chemical potential P for different time instant $t$ and distance $x$.

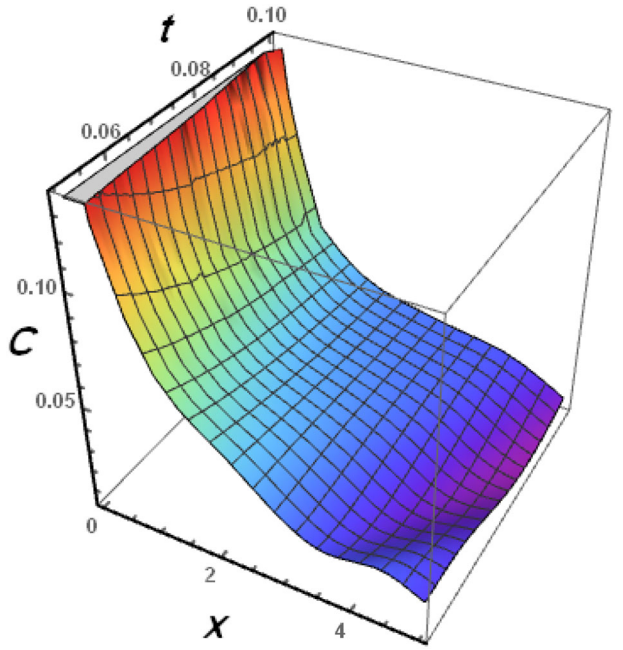


**Figure 5.** The concentration $C(x,t)$ for different time instant $t$ and distance $x$.

- The well-posedness question of the relevant thermoelastic model with time differential phase-lags has been lectured by D'Apice, Chiriţă, and Zampoli (2016) and Zampoli and Landi (2017). Thus, we can say that the current work promotes the validity of the mathematical model with higher-order time-derivatives in the context of the fractional thermo-diffusion theory.

### 7.2. The effect of instant time on thermal and diffusion vibration

Figures 1–5 show 3D plots that are displayed to explain comparisons of the variations of temperature $\theta$, displacement $u$, stress $\sigma_{xx}$, chemical potential $P$ and concentration $C$ versus instant time parameter $t$, $(0.05 \leq t \leq 0.1)$ and the distance $x$, $(0 \leq x \leq 5)$. The distributions are obtained in the frame of the fractional thermo-diffusion theory with higher-order fractional time-derivatives and four phase-lags (HFTD). Numerical results and graphs are obtained when $l = 2, m = 2, n = 2$, $h = 2$ and $\alpha = 0.7$. From the figures, we observed that:

- The instant time $t$ has a significant effect on all the physical studied fields.
- The displacement $u$ and chemical potential $P$ are directly decreasing as the time $t$ increases.
- The chemical potential $P$ is very sensitive to the variation of the distance $x$ andthe instant time $t$.
- The temperature $\theta$ and stress $\sigma_{xx}$ as well as concentration $C$ are no longer increasing with the variationof time and have the same values at the same distance.

## 8. Conclusions

In this paper, we have proposed generalizations of the Fourier law in the theory of heat conduction and of the Fick equation in the diffusion theory. We formulated the theory of thermo-diffusion in deformable bodies, based on the equation of heat conduction and the equation of diffusion with higher-order fractional time-derivatives and four phase-lags of the heat flux vector, temperature gradient, diffusing mass vector and for the chemical potential gradient. Several particular cases of interest have been deduced from the current investigation. Validation results indicate that:

- The new thermo-diffusion model of higher-order fractional time-derivatives and four phase-lags not only matches the experimental data well but can also be a good alternative to the previously used models.
- According to the higher-order thermo-diffusion, we have to construct a new class of materials according to the choice of the values of the parameters $l, m, n$ and $h$. Therefore, these parameters become a new indicator of their ability to conduct heat in conducting materials.
- Some results have been reported to show the difference between classical and modified models of thermo-diffusion theory.
- Moreover, it should be noted that our choices of the expansion orders parameters $l$, $m$, $n$ and $h$ not only correspond to what has been demonstrated in Chiriță, Ciarletta, and Tibullo (2017) but are also able to comprise different thermo-diffusion modes.
- Finally, the results of this work may prove to be useful in the various internal heat-generation problems like nuclear boiling, phonon-electron interactions, exothermic catalytic reactions and phonon scattering that had different applications in geophysical and industrial.

## Nomenclature

| | |
|---|---|
| $\lambda,\ \mu$ | Lamé's constants |
| $\alpha_t$ | thermal expansion coefficient |
| $\alpha_c$ | coefficient of linear diffusion |
| $\beta_1 = (3\lambda + 2\mu)\alpha_t$ | thermal coupling parameter |
| $T_0$ | environmental temperature |
| $\theta = T - T_0$ | temperature increment |
| $T$ | absolute temperature |
| $C_e$ | specific heat |
| $e = \mathrm{div}\mathbf{u}$ | cubical dilatation |
| $\sigma_{ij}$ | stress tensor |

| | |
|---|---|
| $e_{ij}$ | strain tensor |
| $\boldsymbol{u}$ | displacement vector |
| $\boldsymbol{q}$ | heat flux vector |
| $\boldsymbol{\eta}$ | flow of diffusing mass vector |
| $l, m, n, h$ | higher order fractional time derivatives |
| $F_i$ | components of the body force vector |
| $K$ | thermal conductivity |
| $\rho$ | material density |
| $Q$ | heat source |
| $\beta_2 = (3\lambda + 2\mu)\alpha_c$ | diffusion coupling parameter |
| $\delta_{ij}$ | Kronecker′s delta function |
| $\nabla^2$ | Laplacian operator |
| $\tau_q$ | phase lag of heat flux |
| $\tau_\theta$ | phase lag of temperature gradient |
| $\tau_\eta$ | phase lag of diffusing mass |
| $\tau_p$ | phase lag of chemical potential gradient |
| $\alpha$ | fractional order |
| $D$ | diffusion coefficient |
| $P$ | chemical potential |
| $C$ | concentration of diffusion material |
| $a$ | thermoelastic diffusion effect |
| $b$ | measure of diffusive effect |

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